Oral exam, applied and computational mathematics, individual, 2013

Problem 1. Suppose $A \in R^{n\times n}$ is nonsingular, u and v are two vectors.

(a) Find condition such that $A + uv^\top$ is invertible, in that case find $(A + uv^\top)^{-1}$.

(b) Change the first column of A: $\begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{pmatrix}$ by $\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$ resulting in a new matrix $\bar{A}$. Find $(\bar{A})^{-1}$.

Problem 2. A discrete surface is represented as a simplicial complex, such that each face is a Euclidean triangle, which is also called a triangle mesh. Suppose $M = (V, E, F)$ is a triangle mesh, where V, E, F represents the set of vertices, edges and faces respectively. The Euler number of M is defined as

$$\chi(M) := |V| + |F| - |E|.$$

An edge is called an interior edge if it is adjacent to two faces; an edge is called a boundary edge if it is adjacent to only one face. A vertex is called an interior vertex, if all the edges adjacent to it are interior; a vertex is called a boundary vertex, if it attaches to at least one boundary edge. A triangle mesh is called a closed mesh, if it has no boundary edges.

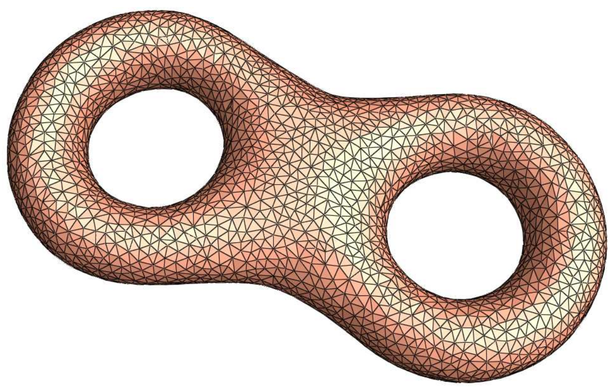

Figure 1: A discrete surface is represented as a triangle mesh.

Suppose $v_i \in V$ is an interior vertex on M, $[v_i, v_j, v_k] \in F$ is a face on M. θ_i^{jk} is the corner angle on the face $[v_i, v_j, v_k]$ with apex v_i. Then the discrete Gaussian curvature at v_i is defined as

$$K(v_i) := 2\pi - \sum_{[v_i,v_j,v_k]\in F} \theta_i^{jk}.$$

If v_i is an boundary vertex, then the discrete Gaussian curvature at v_i is defined as

$$K(v_i) := \pi - \sum_{[v_i,v_j,v_k]\in F} \theta_i^{jk}.$$

1. Suppose M is a closed triangle mesh, prove the Discrete Gauss-Bonnet theorem:

$$\sum_{v_i\in V} K(v_i) = 2\pi\chi(M).$$

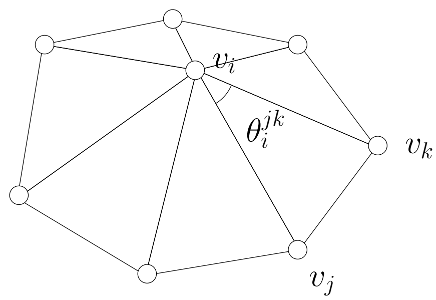

Figure 2: Discrete Gaussian curvature for an interior vertex.

2. (optional) Suppose the faces of M are not only triangles, but also general planar polygons, prove the discrete Gauss-Bonnet theorem.
3. (Optional) Suppose M is a triangle mesh with boundaries, different boundary connect components have no intersection, prove the discrete Gauss-Bonnet theorem.